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Finite-size scaling functions and conformal invariance

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Abstract. A method to calculate perturbative corrections to the spectrum of a one-dimensional quantum chain in the scaling variable z is developed from the conformal invariance at its fixed point. Then, from the knowledge of certain correlation functions, one can calculate the first coefficients of the scaling functions. The method is applied successfully to the Ising model, where the corrections are also known from the exact solution. The first coefficients of the magnetic scaling functions of the Ising model are also obtained.

1. Introduction

In this paper we consider one-dimensional quantum chains, with an infinite number of sites N. Suppose the Hamiltonian of such a quantum chain to be conformally invariant at the critical point. If one succeeds in determining the central charge of this system—we restrict our considerations to $c<1$—then all possible (not every representation has to be realised) energy eigenvalues are known at the critical point (Belavin *et al* 1984, Friedan *et al* 1984). The aim of this paper is to present a perturbative method to calculate corrections to the conformal spectrum due to an external field, i.e. to obtain the scaling functions perturbatively.

Let us first review some known results. The spectrum of a quantum chain at the critical point in the finite-size scaling limit is given by certain products of two irreducible representations (IR) Δ and $\bar{\Delta}$ of two commuting Virasoro algebras with the same central charge c (Friedan *et al* 1984). We denote by Δ the highest weight, and by $\Delta+r$ the rth level having degeneracy $d(\Delta, r)$ of one IR of the Virasoro algebra. (The degeneracies $d(\Delta, r)$ can be computed using the character formulae of Rocha-Caridi (1985).) A state will be labelled by $|\Delta+r, \bar{\Delta}+\bar{r}; i\rangle$ ($i=1,2,\ldots, d(\Delta, r)d(\bar{\Delta}, \bar{r})$). The Hamiltonian is given by

$$H^{c}=\frac{1}{2\pi}\int_{-N/2}^{N/2} dv(T(w)+\bar{T}(\bar{w}))+\text{regular terms} \tag{1.1}$$

where the limit $N\to\infty$ is understood, T is the stress tensor and $w=\tau+iv$ ($-\infty<\tau<\infty, -N/2<v<N/2$) is a variable on the strip ($\bar{w}=\tau-iv$). (We choose periodic or twisted boundary conditions on the strip.) The scaled energy gaps are given by

$$\mathscr{F}(\Delta+r, \bar{\Delta}+\bar{r}; i)=\lim_{N\to\infty}\frac{N}{2\pi}(E(\Delta+r, \bar{\Delta}+\bar{r}; i)-E(0,0))=\Delta+r+\bar{\Delta}+\bar{r} \tag{1.2}$$

where we omit the index i if the state is non-degenerate, and E denotes the energy.

Now we are able to state the problem. If we introduce an external field p (for p one can choose, for example, the reduced temperature $t=(T-T_c)/T_c$ or the magnetic field h) some local fields $\phi_j(w,\bar{w})$ of the conformal theory will couple to p, so that

$$H=H^c+p\int_{-N/2}^{N/2}\mathrm{d}v\sum_j a_p^j\phi_j(0,v) \tag{1.3}$$

where a_p^j are unknown constants and ϕ_j has scaling dimensions $(\Delta_j+r_j, \bar{\Delta}_j+\bar{r}_j)$. The choice $\tau=0$ is arbitrary due to translation invariance in the τ direction. (Sometimes we write $\phi(\tau,v)$ instead of $\phi(w,\bar{w})$.) As will be shown the operator $pa_p^j\int_{-N/2}^{N/2}\mathrm{d}v\,\phi_j(0,v)$ gives, in k order, corrections proportional to

$$\left[2\pi pa_p^j\left(\frac{N}{2\pi}\right)^{2-x_j}\right]^k \tag{1.4}$$

where $x_j=\Delta_j+r_j+\bar{\Delta}_j+\bar{r}_j$. Let

$$x_p=\min_{j(a_p^j\neq 0)} x_j \tag{1.5}$$

and consider the limit $p\to 0$, $N\to\infty$, $z_p=pN^{2-x_p}$ fixed. In this limit only fields ϕ_j with

$$x_j=x_p<2 \tag{1.6}$$

will survive. From this we see that r_j and $\bar{r}_j$ can only take the values 0 or 1. (Notice that in general one can have more restrictions on ϕ_j, since only certain symmetries are broken by choosing z_p different from zero.)

Suppose that for a given p, there exists only one field ϕ_j satisfying these conditions. Then for $x_p<1$ (for $x_p\geqslant 1$ one obtains in general ultraviolet divergent integrals) and z_p small enough one obtains for the scaled energy gaps of the non-degenerate levels the Privman–Fisher (1984) universality hypothesis, namely

$$\mathscr{F}(\Delta+r,\bar{\Delta}+\bar{r})=S^p_{\Delta+r,\bar{\Delta}+\bar{r}}(a_p^j z_p) \tag{1.7}$$

where $S^p_{\Delta+r,\bar{\Delta}+\bar{r}}$ is a universal function (the constant a_p^j is non-universal). Consider, for example, the universality class of the tricritical Ising model, which has the thermal exponent $x_t=\frac{1}{5}$ and the magnetic exponent $x_h=\frac{3}{40}$ (Wu 1982 and references therein). Its fixed point Hamiltonian is conformally invariant with central charge $c=\frac{7}{10}$ (Dotsenko 1984). Looking at the possible anomalous dimensions one finds for corrections in z_t and z_h only one solution of (1.6), namely $r_j=\bar{r}_j=0$, $\Delta_j=\bar{\Delta}_j=x_t/2$ and $x_h/2$, respectively. This is also the case for the Ising model ($c=\frac{1}{2}$, for $x=1$ the integrals for the energy *gaps* are convergent), the three-state Potts model ($c=\frac{4}{5}$) and the tricritical three-state Potts model ($c=\frac{6}{7}$), so that the Privman–Fisher universality hypothesis holds in all four cases for the thermal scaling variable z_t and the magnetic one z_h.

The paper is organised as follows. In § 2 we present a method—which requires the knowledge of certain correlation functions—to calculate corrections to the energy gaps in z_p. We should stress that the method has been developed only for non-degenerate states. In § 3 we calculate corrections in z_t to the energy gaps of the Ising model ($c=\frac{1}{2}$) up to $O(z_t^6)$ achieving full agreement with the results of Henkel (1987), obtained by explicit expansions of the exact solution. For periodic boundary conditions we also calculate the first corrections in z_h, which were not known. A summary of our results is given in § 4. In the appendix we give the correlation functions of the Ising model that are needed to calculate corrections in z_t in any order.

2. Perturbation theory using conformal invariance

In this section we develop a method to calculate corrections in the scaling variable z (from now on we omit the index p) to the non-degenerate levels of the conformal spectrum. Consider a strip of width N with periodic or twisted boundary conditions and let

$$H = H^c + azN^{x-2}\int_{-N/2}^{N/2} \mathrm{d}v\, \phi(0, v) = H^c + V \tag{2.1}$$

where the limit $N\to\infty$, z fixed is understood, ϕ has dimensions $(\Delta_1 + r_1, \bar{\Delta}_1 + \bar{r}_1)$ and $x = \Delta_1 + r_1 + \bar{\Delta}_1 + \bar{r}_1 < 2$. The generalisation to the case where the corrections are due to more than one local field will become obvious.

Let $|p, s\rangle$ $(s = 0, 1, 2, \ldots)$ denote the states with momentum p and energy $E_s(p)$ so that $E_s(p) \leq E_{s+1}(p)$. We have $|\Delta + r, \bar{\Delta} + \bar{r}\rangle = |q, n\rangle$, where $q = \Delta + r - \bar{\Delta} - \bar{r}$ and n is fixed. In order to calculate corrections to the energy of this state, one has only to consider states $|q, s\rangle$, since the v integration in (2.1) acts as a projector.

First we need to know the functions

$$\begin{aligned}\langle q, n|\phi(w_1, \bar{w}_1)\ldots\phi(w_k, \bar{w}_k)|q, n\rangle \\ = \left(\frac{2\pi}{N}\right)^{kx} \sum_{\substack{\nu_1,\ldots,\nu_{k-1}\\ \bar{\nu}_1,\ldots,\bar{\nu}_{k-1}}} \alpha^{(q,n)}_{\nu_1,\ldots,\nu_{k-1};\bar{\nu}_1,\ldots,\bar{\nu}_{k-1}} \xi_1^{\nu_1}\ldots\xi_{k-1}^{\nu_{k-1}}\bar{\xi}_1^{\bar{\nu}_1}\ldots\bar{\xi}_{k-1}^{\bar{\nu}_{k-1}}\end{aligned} \tag{2.2}$$

where $\xi_i = \exp(2\pi/N)(w_i - w_{i+1})$. Notice that the summation indices need not to be integers ($\nu_i - \bar{\nu}_i$ are integers) and that the coefficients $\alpha^{(q,n)}$ are N independent.

In order to obtain the functions (2.2) consider first the case where Δ and $\bar{\Delta}$ are different from zero ($|q, n\rangle \equiv |\Delta + r, \bar{\Delta} + \bar{r}\rangle$). Expanding the $(k+2)$-point function

$$\begin{aligned}\langle\phi_{\Delta,\bar{\Delta}}(w_0, \bar{w}_0)\phi(w_1, \bar{w}_1)\ldots\phi(w_k, \bar{w}_k)\phi_{\Delta,\bar{\Delta}}(w_{k+1}, \bar{w}_{k+1})\rangle \\ = \left(\frac{2\pi}{N}\right)^{kx+2(\Delta+\bar{\Delta})} \sum_{\substack{\nu_0,\ldots,\nu_k\\ \bar{\nu}_0,\ldots,\bar{\nu}_k}} a_{\nu_0,\ldots,\nu_k;\bar{\nu}_0,\ldots,\bar{\nu}_k} \xi_0^{\nu_0}\ldots\xi_k^{\nu_k}\bar{\xi}_0^{\bar{\nu}_0}\ldots\bar{\xi}_k^{\bar{\nu}_k}\end{aligned} \tag{2.3}$$

and the two-point function

$$\begin{aligned}\langle\phi_{\Delta,\bar{\Delta}}(w_0, \bar{w}_0)\phi_{\Delta,\bar{\Delta}}(w_{k+1}, \bar{w}_{k+1})\rangle \\ = \left(\frac{2\pi}{N}\right)^{2(\Delta+\bar{\Delta})} \sum_{\nu,\bar{\nu}} b_{\nu,\bar{\nu}}(\xi_0\ldots\xi_k)^{\nu}(\bar{\xi}_0\ldots\bar{\xi}_k)^{\bar{\nu}}\end{aligned} \tag{2.4}$$

one can show using the spectral decomposition (the state $|q, n\rangle$ is supposed to be non-degenerate) that

$$\alpha^{(q,n)}_{\nu_1,\ldots,\nu_{k-1};\bar{\nu}_1,\ldots,\bar{\nu}_{k-1}} = (b_{\Delta+r;\bar{\Delta}+\bar{r}})^{-1} a_{\Delta+r,\Delta+r+\nu_1,\ldots,\Delta+r+\nu_{k-1},\Delta+r;\bar{\Delta}+\bar{r},\bar{\Delta}+\bar{r}+\bar{\nu}_1,\ldots,\bar{\Delta}+\bar{r}}. \tag{2.5}$$

If Δ and $\bar{\Delta}$ are zero one has to replace $\phi_{\Delta,\bar{\Delta}}(w, \bar{w})$ by $T(w)\bar{T}(\bar{w})$ in (2.3). (In the appendix we remind the reader how to compute such correlation functions from those containing no T. If, say $\Delta = 0$ and $\bar{\Delta} \neq 0$, one has to treat the w and $\bar{w}$ dependence separately.) Choosing

$$w_j = (\tau_2 + \tau_3 + \ldots + \tau_j) + \mathrm{i}v_j \tag{2.6}$$

we obtain

$$\int_{-N/2}^{N/2} \mathrm{d}v_1 \ldots \int_{-N/2}^{N/2} \mathrm{d}v_k \langle q, n|\phi(w_1, \bar{w}_1) \ldots \phi(w_k, \bar{w}_k)|q, n\rangle$$
$$= \left(\frac{2\pi}{N}\right)^{kx} N^k \sum_{\mu_2,\ldots,\mu_k} \beta^{(q,n)}_{\mu_2,\ldots,\mu_k} \exp\left(-2\frac{2\pi}{N} \sum_{i=2}^{k} \mu_i \tau_i\right) \tag{2.7}$$

where $\beta^{(q,n)}_{\mu_2,\ldots,\mu_k} = \alpha^{(q,n)}_{\mu_2,\ldots,\mu_k;\mu_2,\ldots,\mu_k}$. For convenience, we introduce

$$B^{(p_2,\ldots,p_k)}_{q,n;k}(\tau_2, \ldots, \tau_k) = \sum_{\mu_2,\ldots,\mu_k \neq 0} \mathrm{sgn}(\mu_2^{p_2} \ldots \mu_k^{p_k}) \beta^{(q,n)}_{\mu_2,\ldots,\mu_k} \exp\left(-2 \sum_{i=2}^{k} |\mu_i| \tau_i\right) \tag{2.8}$$

where p_i are positive integers.

Now we are able to list the first orders in perturbation theory. Since the state $|q, n\rangle$ is supposed to be non-degenerate, we have (V is given in (2.1))

$$E^{(1)}_n(q) = \langle q, n| V|q, n\rangle = \frac{(2\pi)^x}{N} azB_{q,n;1} \tag{2.9}$$

and consequently

$$\mathscr{F}^{(1)}(\Delta + r, \bar{\Delta} + \bar{r}) = (2\pi)^{x-1} az(B_{q,n;1} - B_{0,0;1}). \tag{2.10}$$

Consider the second-order corrections

$$E^{(2)}_n(q) = -\sum_{i\neq n} \frac{\langle q, n| V|q, i\rangle\langle q, i| V|q, n\rangle}{\omega_{in}}$$
$$= -(azN^{x-2})^2 \int_0^\infty \mathrm{d}\tau_2 \int_{-N/2}^{N/2} \mathrm{d}v_1 \int_{-N/2}^{N/2} \mathrm{d}v_2 \Bigg(\sum_i \exp(-\omega_{in}\tau_2)$$
$$\times \langle q, n|\phi(0, v_1)|q, i\rangle\langle q, i|\phi(0, v_2)|q, n\rangle$$
$$-\langle q, n|\phi(0, v_1)|q, n\rangle\langle q, n|\phi(0, v_2)|q, n\rangle$$
$$- \sum_{i<n} [\exp(-\omega_{in}\tau_2) + \exp(\omega_{in}\tau_2)]\langle q, n|\phi(0, v_1)|q, i\rangle\langle q, i|\phi(0, v_2)|q, n\rangle\Bigg) \tag{2.11}$$

where $\omega_{in} = E^{(0)}_i - E^{(0)}_n$. Here we have used $a^{-1} = \int_0^\infty \exp(-ax)\,\mathrm{d}x$ for $a > 0$. Using the spectral decomposition, one obtains

$$\int_{-N/2}^{N/2} \mathrm{d}v_2 \sum_i \langle q, n|\phi(0, v_1)|q, i\rangle\langle q, i|\phi(0, v_2)|q, n\rangle \exp(-\omega_{in}\tau_2)$$
$$= \int_{-N/2}^{N/2} \mathrm{d}v_2 \langle q, n|\phi(0, v_1)\phi(\tau_2, v_2)|q, n\rangle. \tag{2.12}$$

Inserting this into (2.11) and using (2.8) one obtains

$$E^{(2)}_n(q) = -\left(\frac{(2\pi)^x}{N} az\right)^2 \int_0^\infty \mathrm{d}\tau_2\, B^{(1)}_{q,n;2}\left(\frac{2\pi}{N}\tau_2\right) \tag{2.13}$$

and. consequently,

$$\mathscr{F}^{(2)}(\Delta + r, \bar{\Delta} + \bar{r}) = -[(2\pi)^{x-1} az]^2 \int_0^\infty \mathrm{d}\tau_2 (B^{(1)}_{q,n;2}(\tau_2) - B^{(1)}_{0,0;2}(\tau_2)). \tag{2.14}$$

For higher orders this can be generalised easily, supposing the standard formulae of perturbation theory are known. Here we give the results for the third and fourth order.

From

$$E_n^{(3)} = \sum_{i\neq n}\sum_{j\neq n} \frac{\langle n|V|i\rangle\langle i|V|j\rangle\langle j|V|n\rangle}{\omega_{in}\omega_{jn}} - \langle n|V|n\rangle \sum_{i\neq n} \frac{\langle n|V|i\rangle\langle i|V|n\rangle}{\omega_{in}^2} \tag{2.15}$$

one obtains

$$\mathscr{F}^{(3)}(\Delta+r, \bar\Delta+\bar r) = [(2\pi)^{x-1}az]^3 \int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_3 \{[B^{(1,1)}_{q,n;3}(\tau_2,\tau_3) - B_{q,n;1}B^{(2)}_{q,n;2}(\tau_2+\tau_3)] - [(q,n)\leftrightarrow(0,0)]\} \tag{2.16}$$

and from

$$\begin{aligned} E_n^{(4)} = &-\sum_{i\neq n}\sum_{j\neq n}\sum_{k\neq n} \frac{\langle n|V|i\rangle\langle i|V|j\rangle\langle j|V|k\rangle\langle k|V|n\rangle}{\omega_{in}\omega_{jn}\omega_{kn}} \\ &+\langle n|V|n\rangle \sum_{i\neq n}\sum_{j\neq n} \frac{\langle n|V|i\rangle\langle i|V|j\rangle\langle j|V|n\rangle}{\omega_{in}\omega_{jn}}\left(\frac{1}{\omega_{in}}+\frac{1}{\omega_{jn}}\right) \\ &+\sum_{i\neq n}\frac{\langle n|V|i\rangle\langle i|V|n\rangle}{\omega_{in}^2}\sum_{j\neq n}\frac{\langle n|V|j\rangle\langle j|V|n\rangle}{\omega_{jn}} \\ &-(\langle n|V|n\rangle)^2\sum_{i\neq n}\frac{\langle n|V|i\rangle\langle i|V|n\rangle}{\omega_{in}^3} \end{aligned} \tag{2.17}$$

one obtains

$$\begin{aligned} \mathscr{F}^{(4)}(\Delta+r, \bar\Delta+\bar r) = -[(2\pi)^{x-1}az]^4 \int_0^\infty \mathrm{d}\tau_2 \int_0^\infty \mathrm{d}\tau_3 \int_0^\infty \mathrm{d}\tau_4 (\{B^{(1,1,1)}_{q,n;4}(\tau_2,\tau_3,\tau_4) \\ - B_{q,n;1}[B^{(2,1)}_{q,n;3}(\tau_2+\tau_3,\tau_4) + B^{(1,2)}_{q,n;3}(\tau_2,\tau_3+\tau_4)] \\ - B^{(2)}_{q,n;2}(\tau_2+\tau_3)B^{(1)}_{q,n;2}(\tau_4) + (B_{q,n;1})^2 B^{(3)}_{q,n;2}(\tau_2+\tau_3+\tau_4)\} \\ - [(q,n)\leftrightarrow(0,0)]) \end{aligned} \tag{2.18}$$

where in the standard formulae (2.15) and (2.17) the index q has been omitted.

Let us summarise. If the $(k+2)$-point functions $\langle\phi_{\Delta,\bar\Delta}(w_0,\bar w_0)\phi(w_1,\bar w_1)\ldots\phi(w_k,\bar w_k)\phi_{\Delta,\bar\Delta}(w_{k+1},\bar w_{k+1})\rangle$ are known for every primary field $\phi_{\Delta,\bar\Delta}$ of the conformal theory, then one can calculate the corrections to the non-degenerate spectrum of the conformal theory due to the perturbation (2.1) up to the kth order.

3. Application to the Ising model

In this section we calculate corrections in z_t and z_h to the universality class of the Ising model using our previous results and compare the corrections in z_t to the results of Henkel (1987) obtained from the exact solution for $h=0$. For the sake of completeness, we give the Hamilton operator of this universality class, defined on a chain with N sites

$$H = -\frac{\lambda}{2\gamma}\sum_{i=1}^N \sigma^z(i) - \frac{1}{4\gamma}\sum_{i=1}^N [(1+\gamma)\sigma^x(i+1)\sigma^x(i) + (1-\gamma)\sigma^y(i+1)\sigma^y(i)] - \frac{h}{2\gamma}\sum_{i=1}^N \sigma^x(i) \tag{3.1}$$

where σ^x, σ^y, σ^z are the Pauli matrices and λ has the meaning of an inverse temperature. For $h=0$ and all γ $(0<\gamma\leq 1)$, there is a critical point at $\lambda_c=1$ $(N=\infty)$, which can be described by the IR of two Virasoro algebras with central charge $c=\frac{1}{2}$. Choosing periodic or antiperiodic boundary conditions in (3.1)

$$\sigma(N+1)=(-1)^{\tilde{Q}}\sigma(1) \qquad \tilde{Q}=0,1 \tag{3.2}$$

one obtains the Hamiltonians $H^{(\tilde{Q})}$. Since H commutes for $h=0$ with the charge operator

$$Q=\frac{1}{2}\left(\mathbb{1}-\prod_{i=1}^{N}\sigma^z(i)\right) \tag{3.3}$$

which has the eigenvalues 0 and 1, one is left with the blocks $H_Q^{(\tilde{Q})}$. It was shown (Cardy 1986, Henkel 1987) that the corresponding spectra $\mathscr{E}_Q^{(\tilde{Q})}$ at the critical point can be described by the following sums of IR of the Virasoro algebra:

$$\begin{aligned}
&\mathscr{E}_0^{(0)}=(0,0)\oplus(\tfrac{1}{2},\tfrac{1}{2})\\
&\mathscr{E}_0^{(1)}=\mathscr{E}_1^{(0)}=(\tfrac{1}{16},\tfrac{1}{16})\\
&\mathscr{E}_1^{(1)}=(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2}).
\end{aligned} \tag{3.4}$$

The degeneracies $d(\Delta, r)$ are equal to one only in the following cases (Henkel 1987): $d(\frac{1}{16}, r)$, $r=0,1,2$; $d(\frac{1}{2}, r)$, $r=0,1,2,3$; $d(0,r)$, $r=0,2,3$. This is the situation for $\lambda=\lambda_c$, $h=0$ and $N\to\infty$ in (3.1) (i.e. $H=H^c$). Now for $h=0$ consider the finite-size scaling limit $N\to\infty$, $\lambda\to\lambda_c$ $(=1)$ and $z=(1-\lambda)N$ fixed (the Ising model has the thermal exponent $x_t=1$) in (3.1). Then from the previous section we know that

$$H=H^c+az\lim_{N\to\infty}\frac{1}{N}\int_{-N/2}^{N/2}\mathrm{d}v\,\varepsilon(0,v)=H^c+V \tag{3.5}$$

where ε is the energy density $(\Delta=\bar{\Delta}=\frac{1}{2})$ and the constant a is non-universal (i.e. γ dependence). Consider for example the second-order corrections to the state $|r+\frac{1}{16}, \bar{r}+\frac{1}{16}\rangle$. From the correlation functions $\langle\sigma\varepsilon\varepsilon\sigma\rangle$ and $\langle\sigma\sigma\rangle$ (see the appendix) one obtains

$$\begin{aligned}
&\langle r+\tfrac{1}{16}, \bar{r}+\tfrac{1}{16}|\varepsilon(w_1,\bar{w}_1)\varepsilon(w_2,\bar{w}_2)|r+\tfrac{1}{16}, \bar{r}+\tfrac{1}{16}\rangle=\left(\frac{2\pi}{N}\right)^2 f_r(\xi_1)f_{\bar{r}}(\bar{\xi}_1)\\
&f_r(\xi_1)=\tfrac{1}{2}+\xi_1/(1-\xi_1)+\xi_1^{-r}-\xi_1^r
\end{aligned} \tag{3.6}$$

so that

$$\begin{aligned}
B^{(1)}_{(1/16+r,1/16+\bar{r});2}(\tau)&=\exp(-2\tau)/[1-\exp(-2\tau)]\\
&\quad-(1-\delta_{r,0})\exp(-2\tau r)-(1-\delta_{\bar{r},0})\exp(-2\tau\bar{r}).
\end{aligned} \tag{3.7}$$

An analogous calculation for the ground state yields

$$B^{(1)}_{0,0;2}(\tau)=\exp(-\tau)/[1-\exp(-2\tau)] \tag{3.8}$$

so that from (2.14) we have

$$\mathscr{F}^{(2)}(\tfrac{1}{16}+r,\tfrac{1}{16}+\bar{r})=(az)^2[\ln 2+(1-\delta_{r,0})/2r+(1-\delta_{\bar{r},0})/2\bar{r}] \qquad r,\bar{r}=0,1,2. \tag{3.9}$$

We have calculated the first five orders in perturbation theory to all non-degenerate levels of the conformal spectra. Our results are

$$\begin{aligned}\mathscr{F}(r+\tfrac{1}{16}, \bar{r}+\tfrac{1}{16}) &= (r+\bar{r}+\tfrac{1}{8}) + az\tfrac{1}{2}(2\delta_{r,0}-1)(2\delta_{\bar{r},0}-1)\\ &\quad + (az)^2[\ln 2 + (1-\delta_{r,0})/2r + (1-\delta_{\bar{r},0})/2\bar{r}]\\ &\quad - (az)^4[\tfrac{3}{4}\zeta(3) + (1-\delta_{r,0})/8r^3 + (1-\delta_{\bar{r},0})/8\bar{r}^3]\\ &\quad + O(z^6) \qquad (r, \bar{r} = 0, 1, 2)\end{aligned}$$

and for Δ and $\bar{\Delta}$ equal to 0 or $\frac{1}{2}$

$$\begin{aligned}\mathscr{F}(r+\Delta, \bar{r}+\bar{\Delta}) &= (r+\Delta+\bar{r}+\bar{\Delta}) + (az)^2(\alpha(r,\Delta)+\alpha(\bar{r},\bar{\Delta}))\\ &\quad + (az)^4(\beta(r,\Delta)+\beta(\bar{r},\bar{\Delta})) + O(z^6)\\ \alpha(r,0) &= 1+1/(2r-1) \qquad \beta(r,0) = 1+1/(2r-1)^3 \qquad r=0,2,3\\ \alpha(r,\tfrac{1}{2}) &= 1/(2r+1) \qquad \beta(r,\tfrac{1}{2}) = 1/(2r+1)^3 \qquad r=0,1,2,3\end{aligned} \tag{3.10}$$

where ζ is the Riemann zeta function.

This reproduces the results of Henkel (1987)—obtained from the exact solution—with

$$a = (-1)^{\tilde{Q}}\,\mathrm{sgn}(z)(1/2\pi\gamma). \tag{3.11}$$

Thus, the coupling constant a has a simple dependence on the boundary conditions ($\tilde{Q}$). The appearance of sgn(z) is not surprising (symmetric corrections for higher and lower temperatures than the critical one).

Let us make one more remark. In the appendix we give the correlation functions which are needed to calculate corrections in z in any order, so that in principle it is possible to obtain the perturbation series (combinatorial problem) and thus the scaling functions.

Finally we give the first expansion coefficients of the magnetic scaling functions. For $\lambda = \lambda_c$ consider the finite-size scaling limit $N \to \infty$, $h \to 0$ and $z_h = hN^{15/8}$ fixed ($x_h = \frac{1}{8}$ for the Ising model) in (3.1) with periodic boundary conditions, so that

$$H = H^c + bz_h \lim_{N\to\infty} N^{1/8-2}\int_{-N/2}^{N/2} \mathrm{d}v\,\sigma(0,v) \tag{3.12}$$

where σ has scaling dimensions $\Delta = \bar{\Delta} = \frac{1}{16}$. Since the correlation functions with an odd number of σ vanish one obtains only even order corrections. Using the methods of the previous section we have obtained the second-order corrections to the scaled spectrum. For the charge sector zero we have

$$\begin{aligned}\mathscr{F}(r+\tfrac{1}{2}, \bar{r}+\tfrac{1}{2}) &= r+\bar{r}+1+[bz_h/(2\pi)^{7/8}]^2\delta + O(z_h^4) \qquad 0\leq r, \bar{r}\leq 3\\ \mathscr{F}(r,\bar{r}) &= r+\bar{r}+[bz_h/(2\pi)^{7/8}]^2(1-\gamma_r\gamma_{\bar{r}})\delta + O(z_h^4) \qquad r,\bar{r}=0,2,3\end{aligned} \tag{3.13}$$

where $\gamma_0 = 1$, $\gamma_2 = \frac{19}{124}$, $\gamma_3 = \frac{512}{1457}$ and

$$\delta = \sum_{\nu=0}^{\infty}\left(\frac{\Gamma(\frac{1}{8}+\nu)}{\Gamma(\frac{1}{8})\nu!}\right)^2 \frac{1}{2\nu+\frac{1}{8}} = 8.009\,492\,725\ldots.$$

Notice that the second-order corrections to the states $|r+\frac{1}{2}, \bar{r}+\frac{1}{2}\rangle$ $(0\leq r, \bar{r}\leq 3)$ vanish.

After a tedious calculation one obtains the second-order corrections in the charge sector one using the four-point function $\langle\sigma\sigma\sigma\sigma\rangle$ (see, e.g., Dotsenko 1984)

$$\mathscr{F}(\tfrac{1}{16}+r, \tfrac{1}{16}+\bar{r}) = r+\bar{r}+\tfrac{1}{8}+[bz_h/(2\pi)^{7/8}]^2(\delta - g_{r\bar{r}}) + O(z_h^4) \qquad r,\bar{r}=0,1,2 \tag{3.14}$$

where $g_{r\bar{r}}=\sum_\nu b_\nu^{(r)}b_\nu^{(\bar{r})}1/(\nu-\frac{1}{8})$ and $b_\nu^{(r)}$ are given by

$$\frac{(1-x)^{3/8}}{(1+x)^{1/8}}=\sum_\nu b_\nu^{(0)}x^\nu \qquad \frac{(1-x)^{3/8}}{(1+x)^{1/8}}\left(\frac{1}{8x^2}-\frac{1}{2x}-\frac{1}{4}-\frac{x}{2}+\frac{x^2}{8}\right)=\sum_\nu b_\nu^{(1)}x^\nu$$

$$\frac{(1-x)^{3/8}}{144(1+x)^{1/8}}\left(\frac{113}{8x^4}-\frac{58}{x^3}-\frac{33}{2x^2}-\frac{325}{x}-\frac{501}{4}-164x+\frac{31}{2}x^2-13x^3+\frac{2433}{8}x^4\right)=\sum_\nu b_\nu^{(2)}x^\nu.$$

A numerical calculation yields

$$\begin{array}{lll} g_{00}=-7.706\,849\,20\,(1) & g_{10}=0.226\,672\,03\,(2) & g_{20}=4.282\,379\,0\,(1) \\ g_{11}=-0.006\,666\,83\,(2) & g_{21}=0.150\,822\,0\,(2) & g_{22}=0.617\,748\,4\,(6) \end{array} \tag{3.15}$$

where the given errors are exact upper and lower bounds.

4. Conclusions

We have presented a method—which uses conformal invariance and needs a knowledge of certain correlation functions—to calculate perturbative corrections to the non-degenerate levels of the conformal theory in the finite-size scaling limit, $N\to\infty$, $p\to 0$ and $z=pN^{2-x}$ fixed, where p is an external field (for example, reduced temperature $(T-T_c)/T_c$, magnetic field h, etc) and x its corresponding exponent $(0<x<2)$. Under certain assumptions—which can be easily tested for a concrete model—this method implies the Privman-Fisher (1984) universality hypothesis for the non-degenerate levels.

The method has been applied to a system belonging to the universality class of the Ising model in the finite-size scaling limit $(h=0)$, $N\to\infty$, $\lambda\to\lambda_c$, $z=(\lambda_c-\lambda)N$ fixed (λ is an inverse temperature), reproducing the scaling functions—which are known from the exact solution for $h=0$—up to $O(z^6)$. Since, in this model, all correlation functions that are needed to calculate corrections in z are known (see the appendix), it should be possible to obtain the perturbative series (combinatorial problem) and thus the scaling functions. As a byproduct we have obtained the correlation functions $\langle\sigma\varepsilon\varepsilon\ldots\varepsilon\sigma\rangle$ of the Ising model. For this model we have also considered the finite-size scaling limit $\lambda=\lambda_c$, $N\to\infty$, $h\to 0$ and $z_h=hN^{15/8}$ fixed, thus obtaining the first expansion coefficients of the magnetic scaling functions.

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Appendix

Let us first make some general remarks. If ϕ_i are primary fields of dimensions $(\Delta_i, \bar{\Delta}_i)$ we have on the strip (for the moment we omit the $\bar{w}$ dependence)

$$\langle T(w_0)(L_{-1}\phi_1)(w_1)\ldots(L_{-1}\phi_l)(w_l)\phi_{l+1}(w_{l+1})\ldots\phi_{n-1}(w_{n-1})T(w_n)\rangle$$
$$=\partial_{w_1}\ldots\partial_{w_l}\langle T(w_0)\phi_1(w_1)\phi_2(w_2)\ldots\phi_{n-1}(w_{n-1})T(w_n)\rangle \tag{A1}$$

where $L_{-1}\phi_i \equiv \phi_{\Delta_i+1,\bar{\Delta}_i}$. If one omits the T this formula holds, too. Generally it is more convenient to give correlation functions on the plane than on the strip. They are related by the transformation

$$w \mapsto z = \exp\left(\frac{2\pi}{N} w\right) \tag{A2}$$

where z is a variable on the plane. Under this transformation we have

$$\begin{aligned} \phi_{\Delta,\bar{\Delta}}(w) &\mapsto \left(\frac{2\pi}{N} z\right)^{\Delta} \phi_{\Delta,\bar{\Delta}}(z) \\ T(w) &\mapsto \left(\frac{2\pi}{N}\right)^2 \left(z^2 T(z) - \frac{c}{24}\right) \end{aligned} \tag{A3}$$

where c is the central charge.

From these formulae and from

$$\begin{aligned} \langle T(z_0)\phi_1(z_1)\ldots\phi_{n-1}(z_{n-1})T(z_n)\rangle \\ = \left[\frac{2}{z_{0n}^2} + \frac{1}{z_{0n}}\partial_{z_n} + \sum_{i=1}^{n-1}\left(\frac{\Delta_i}{z_{0i}^2} + \frac{1}{z_{0i}}\partial_{z_i}\right)\right] \\ \times \left[\sum_{i=1}^{n-1}\left(\frac{\Delta_i}{z_{ni}^2} + \frac{1}{z_{ni}}\partial_{z_i}\right)\langle\phi_1(z_1)\ldots\phi_{n-1}(z_{n-1})\rangle\right] \\ + \frac{c}{2}\frac{1}{z_{0n}^4}\langle\phi_1(z_1)\ldots\phi_{n-1}(z_{n-1})\rangle \qquad (z_{ij} = z_i - z_j) \end{aligned} \tag{A4}$$

one sees that it is sufficient to know certain correlation functions on the plane containing only primary fields in order to compute corrections in a scaling variable z_p to the conformal spectrum. After these well known remarks, let us list all the correlation functions of the Ising model that are needed to calculate the corrections in the scaling variable $z_t = N(T - T_c)/T_c$ ($T \to T_c$, $N \to \infty$) to the non-degenerate levels of its conformal spectra, i.e.

$$\langle\phi(z_1, \bar{z}_1)\varepsilon(z_2, \bar{z}_2)\ldots\varepsilon(z_l, \bar{z}_l)\phi(z_{l+1}, \bar{z}_{l+1})\rangle \qquad l \in \mathbb{N} \tag{A5}$$

where ε is the energy density and ϕ is a primary field of the theory ($\phi = \mathbb{1}$, ψ, $\bar{\psi}$, ε, or σ). Since $\varepsilon(z, \bar{z}) = \psi(z)\bar{\psi}(\bar{z})$, and ψ and $\bar{\psi}$ are free Majorana fields (see, e.g., Belavin *et al* (1984)), one has

$$\langle\bar{\psi}(\bar{z}_1)\ldots\bar{\psi}(\bar{z}_n)\rangle = \overline{\langle\psi(z_1)\ldots\psi(z_n)\rangle}$$

$$\langle\psi(z_1)\ldots\psi(z_{2k+1})\rangle = 0$$

and

$$\langle\psi(z_1)\ldots\psi(z_{2k})\rangle = \sum_{\substack{P \\ p_1<p_3<\cdots<p_{2k-1} \\ p_{2i-1}<p_{2i}}} \operatorname{sgn}(P)[z_{p_1p_2}z_{p_3p_4}\ldots z_{p_{2k-1}}z_{p_{2k}}]^{-1}. \tag{A6}$$

Therefore, if $\phi \in \{\mathbb{1}, \psi, \bar{\psi}, \varepsilon\}$, the correlation functions of (A5) are known.

For $\phi = \sigma$ we have for the two-, three- and four-point functions

$$\langle \sigma(\xi, \bar{\xi})\varepsilon(z_1, \bar{z}_1) \ldots \varepsilon(z_l, \bar{z}_l)\sigma(\eta, \bar{\eta})\rangle = |\langle \sigma(\xi)\varepsilon(z_1) \ldots \varepsilon(z_l)\sigma(\eta)\rangle|^2 \tag{A7}$$

where

$$\langle \sigma(\xi)\sigma(\eta)\rangle = (\xi - \eta)^{-1/8}$$

$$\langle \sigma(\xi)\varepsilon(z_1)\sigma(\eta)\rangle = \frac{1}{\sqrt{2}} \frac{(\xi - \eta)^{(1/2-1/8)}}{(\xi - z_1)^{1/2}(z_1 - \eta)^{1/2}} \tag{A8}$$

and

$$\langle \sigma(\xi)\varepsilon(z_1)\varepsilon(z_2)\sigma(\eta)\rangle = \frac{(\xi - \eta)^{(1-1/8)}}{[(\xi - z_1)(\xi - z_2)(z_1 - \eta)(z_2 - \eta)]^{1/2}} \left(\frac{1}{2} + \frac{(\xi - z_1)(z_2 - \eta)}{(\xi - \eta)(z_1 - z_2)}\right).$$

We will show that for $l > 2$ (A7) remains valid, with

$$\frac{\langle \sigma(\xi)\varepsilon(z_1) \ldots \varepsilon(z_{2k})\sigma(\eta)\rangle}{\langle \sigma(\xi)\sigma(\eta)\rangle} = \sum_{\substack{P \\ p_1 < p_3 < \cdots < p_{2k-1} \\ p_{2i-1} < p_{2i}}} \operatorname{sgn}(P) A_{p_1, p_2} A_{p_3, p_4} \ldots A_{p_{2k-1}, p_{2k}}$$

$$\frac{\langle \sigma(\xi)\varepsilon(z_1) \ldots \varepsilon(z_{2k+1})\sigma(\eta)\rangle}{\langle \sigma(\xi)\sigma(\eta)\rangle} = \sum_{\substack{P \\ p_1 < p_3 < \cdots < p_{2k-1} \\ p_{2i-1} < p_{2i}}} \operatorname{sgn}(P) A_{p_1, p_2} A_{p_3, p_4} \ldots A_{p_{2k-1}, p_{2k}} g_{p_{2k+1}} \tag{A9}$$

where

$$A_{i,j} = \frac{\langle \sigma(\xi)\varepsilon(z_i)\varepsilon(z_j)\sigma(\eta)\rangle}{\langle \sigma(\xi)\sigma(\eta)\rangle} = -A_{j,i}$$

$$g_i = \frac{\langle \sigma(\xi)\varepsilon(z_i)\sigma(\eta)\rangle}{\langle \sigma(\xi)\sigma(\eta)\rangle}. \tag{A10}$$

First of all notice that the correlation functions $\langle \sigma(\xi, \bar{\xi})\varepsilon(z_1, \bar{z}_1) \ldots \varepsilon(z_n, \bar{z}_n)\sigma(\eta, \bar{\eta})\rangle$ given by these expressions are local and symmetric under $(\xi, \bar{\xi}) \leftrightarrow (\eta, \bar{\eta})$ and $(z_i, \bar{z}_i) \leftrightarrow (z_j, \bar{z}_j)$. One can easily show that the expressions (A9) satisfy all cluster properties and that the leading term of every short distance expansion is the correct one.

One can also see that these correlation functions are invariant under the small conformal group (they satisfy the corresponding differential equations). So it is sufficient to show that the following second-order differential equations are satisfied:

$$\left(\sum_{i=1}^{n} \mathscr{L}_{-2}(\xi, z_i) + \mathscr{L}_{-2}(\xi, \eta) - \tfrac{4}{3}\partial_\xi^2\right)\langle \sigma(\xi)\varepsilon(z_1) \ldots \varepsilon(z_n)\sigma(\eta)\rangle = 0 \tag{A11}$$

$$\left(\sum_{i=2}^{n} \mathscr{L}_{-2}(z_1, z_i) + \mathscr{L}_{-2}(z_1, \eta) + \mathscr{L}_{-2}(z_1, \xi) - \tfrac{3}{4}\partial_{z_1}^2\right)\langle \sigma(\xi)\varepsilon(z_1) \ldots \varepsilon(z_n)\sigma(\eta)\rangle = 0 \tag{A12}$$

where

$$\mathscr{L}_{-2}(z, z_i) = \frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i}\partial_{z_i}.$$

Let

$$\tilde{c}_i=\left(\mathscr{L}_{-2}(z_1, z_i)+\frac{1}{z_1-\xi}\partial_\xi+\frac{1}{z_1-\eta}\partial_\eta\right)g_i=\frac{[(\xi-\eta)(\xi-z_i)(z_i-\eta)]^{1/2}}{2\sqrt{2}(\xi-z_1)(z_1-\eta)(z_1-z_i)^2}$$

and

$$c_{i,j}=\left(\mathscr{L}_{-2}(z_1, z_i)+\mathscr{L}_{-2}(z_1, z_j)+\frac{1}{z_1-\xi}\partial_\xi+\frac{1}{z_1-\eta}\partial_\eta\right)A_{i,j}$$
$$=\frac{(z_i-z_j)[(\xi-z_1)^2(z_i-\eta)(z_j-\eta)+(z_1-\eta)^2(\xi-z_i)(\xi-z_j)]}{4[(\xi-z_i)(\xi-z_j)(z_i-\eta)(z_j-\eta)]^{1/2}(\xi-z_1)(z_1-\eta)(z_1-z_i)^2(z_1-z_j)^2}. \tag{A13}$$

Then for $n=2k$ in (A12), we have

$$\left(\sum_{i=2}^{2k}\mathscr{L}_{-2}(z_1, z_i)+\mathscr{L}_{-2}(z_1, \xi)+\mathscr{L}_{-2}(z_1, \eta)-\tfrac{3}{4}\partial_{z_1}^2\right)$$
$$\times\sum_{\substack{P\\ p_1<p_3<\cdots<p_{2k-1}\\ p_{2j-1}<p_{2j}}}\operatorname{sgn}(P)\langle\sigma(\xi)\varepsilon(z_{p_1})\varepsilon(z_{p_2})\sigma(\eta)\rangle\prod_{l=2}^{k}A_{p_{2l-1},p_{2l}}$$
$$=\sum_{\substack{P\\ p_1<p_3<\cdots<p_{2k-1}\\ p_{2j-1}<p_{2j}}}\operatorname{sgn}(P)\langle\sigma(\xi)\varepsilon(z_{p_1})\varepsilon(z_{p_2})\sigma(\eta)\rangle\sum_{i=2}^{k}c_{p_{2i-1},p_{2i}}\prod_{\substack{l=2\\ l\neq i}}^{k}A_{p_{2l-1},p_{2l}}$$
$$=\sum_{2\leqslant i<j<l\leqslant 2k}(-1)^{i+j+l+1}[A_{1,i}C_{j,l}-A_{1,j}C_{i,l}+A_{1,l}C_{i,j}]$$
$$\times\langle\sigma(\xi)\varepsilon(z_2)\varepsilon(z_3)\ldots\widehat{\varepsilon(z_i)}\ldots\widehat{\varepsilon(z_j)}\ldots\widehat{\varepsilon(z_l)}\ldots\varepsilon(z_{2k})\sigma(\eta)\rangle=0 \tag{A14}$$

where we have used the fact that the correlation function $\langle\sigma(\xi)\varepsilon(z_1)\varepsilon(z_{p_2})\sigma(\eta)\rangle$ satisfies the corresponding differential equation and where an explicit calculation shows that the expression in square brackets vanishes. Using this result we have for $n=2k+1$ in (A12) (the differential equation for $\langle\sigma(\xi)\varepsilon(z_1)\sigma(\eta)\rangle$ is satisfied)

$$\left(\sum_{i=2}^{2k+1}\mathscr{L}_{-2}(z_1, z_i)+\mathscr{L}_{-2}(z_1, \xi)+\mathscr{L}_{-2}(z_1, \eta)-\tfrac{3}{4}\partial_{z_1}^2\right)\langle\sigma(\xi)\varepsilon(z_1)\ldots\varepsilon(z_{2k+1})\sigma(\eta)\rangle$$
$$=\sum_{i=2}^{2k+1}(-1)^{i+1}\tilde{c}_i\langle\sigma(\xi)\varepsilon(z_1)\ldots\widehat{\varepsilon(z_i)}\ldots\varepsilon(z_{2k+1})\sigma(\eta)\rangle$$
$$+\sum_{\substack{P\\ p_1=1,\,p_{2i}<p_{2i+1}\\ p_2<p_4<\cdots<p_{2k}}}\operatorname{sgn}(P)\langle\sigma(\xi)\varepsilon(z_1)\sigma(\eta)\rangle\sum_{j=1}^{k}C_{p_{2j},p_{2j+1}}\prod_{\substack{l=1\\ l\neq j}}^{k}A_{p_{2l},p_{2l+1}}$$
$$=\sum_{2\leqslant i<j\leqslant 2k+1}(-1)^{i+j+1}[A_{1,i}\tilde{C}_j-A_{1,j}\tilde{C}_i+C_{i,j}g_1]$$
$$\times\langle\sigma(\xi)\varepsilon(z_2)\ldots\widehat{\varepsilon(z_i)}\ldots\widehat{\varepsilon(z_j)}\ldots\varepsilon(z_{2k+1})\sigma(\eta)\rangle=0 \tag{A15}$$

since the expression in brackets vanishes.

Let

$$B_i=\langle\sigma(\xi)\varepsilon(z_1)\varepsilon(z_2)\ldots\widehat{\varepsilon(z_i)}\ldots\varepsilon(z_n)\sigma(\eta)\rangle. \tag{A16}$$

Equation (A11) is obtained by induction in k. For $n=2k$ we have

$$\left(\mathscr{L}_{-2}(\xi,\eta)-\tfrac{4}{3}\partial_\xi^2+\sum_{j=1}^{2k}\mathscr{L}_{-2}(\xi,z_j)\right)\left(\sum_{i=2}^{2k}(-1)^i A_{1,i}B_i\right)$$

$$=\sum_{i=2}^{2k}(-1)^i\left[B_i\left(\mathscr{L}_{-2}(\xi,z_1)+\mathscr{L}_{-2}(\xi,z_i)-\tfrac{4}{3}\partial_\xi^2+\frac{1}{\xi-\eta}\partial_\eta\right)\right.$$

$$\left.\times A_{1,i}-\tfrac{8}{3}(\partial_\xi A_{1,i})(\partial_\xi B_i)\right]$$

$$=-\tfrac{8}{3}\sum_{i=2}^{2k}(-1)^i\langle\sigma(\xi)\sigma(\eta)\rangle(\partial_\xi A_{1,i})\partial_\xi\left(\frac{B_i}{\langle\sigma(\xi)\sigma(\eta)\rangle}\right)$$

$$=-\tfrac{8}{3}\sum_{\substack{P\\ p_1<p_3<\cdots<p_{2k-1}\\ p_{2j-1}<p_{2j}}}\operatorname{sgn}(P)(\partial_\xi A_{p_1,p_2})\partial_\xi\left(\prod_{l=2}^{k}A_{p_{2l-1},p_{2l}}\right)\langle\sigma(\xi)\sigma(\eta)\rangle$$

$$=-\tfrac{8}{3}\sum_{2\leq i<j<l\leq 2k}(-1)^{i+j+l+1}[(\partial_\xi A_{1,i})(\partial_\xi A_{j,l})$$

$$-(\partial_\xi A_{1,j})(\partial_\xi A_{i,l})+(\partial_\xi A_{1,l})(\partial_\xi A_{i,j})]$$

$$\times\langle\sigma(\xi)\varepsilon(z_2)\varepsilon(z_3)\ldots\widehat{\varepsilon(z_i)}\ldots\widehat{\varepsilon(z_j)}\ldots\widehat{\varepsilon(z_l)}\ldots\varepsilon(z_{2k})\sigma(\eta)\rangle=0 \qquad \text{(A17)}$$

since the expression in brackets vanishes.

For $n=2k+1$ one obtains in a similar way

$$\left(\mathscr{L}_{-2}(\xi,\eta)-\tfrac{4}{3}\partial_\xi^2+\sum_{j=1}^{2k+1}\mathscr{L}_{-2}(\xi,z_j)\right)\left(\sum_{i=1}^{2k+1}(-1)^{i+1}g_iB_i\right)$$

$$=-\tfrac{8}{3}\sum_{i=1}^{2k+1}(-1)^{i+1}\langle\sigma(\xi)\sigma(\eta)\rangle(\partial_\xi g_i)\partial_\xi\left(\frac{B_i}{\langle\sigma(\xi)\sigma(\eta)\rangle}\right)$$

$$=-\tfrac{8}{3}\sum_{1\leq i<j<l\leq 2k+1}(-1)^{i+j+l}[(\partial_\xi g_i)(\partial_\xi A_{j,l})-(\partial_\xi g_j)(\partial_\xi A_{i,l})+(\partial_\xi g_l)(\partial_\xi A_{i,j})]$$

$$\times\langle\sigma(\xi)\varepsilon(z_1)\varepsilon(z_2)\ldots\widehat{\varepsilon(z_i)}\ldots\widehat{\varepsilon(z_j)}\ldots\widehat{\varepsilon(z_l)}\ldots\varepsilon(z_{2k+1})\sigma(\eta)\rangle=0. \qquad \text{(A18)}$$

The expression in brackets vanishes here, too.

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